Fei Qi

Office Hours: Tuesday 5-7PM, Hill 624.
or by appointment.

Email: fq15@scarletmail.rutgers.edu

Website: http://math.rutgers.edu/~fq15

Reference:

W. Rudin. Principle of Mathematical Analysis (classical)

V. Zorich. Mathematical Analysis (modern)

A. Mattuck. Introduction to Analysis (easy, wordy)

Polya-Szegö. Problems and Theorems in Mathematical Analysis.

Recall: A map $f: A \to B$ assigns a unique element $f(a) \in B$ for each $a \in A$.

Example: $A = \{$ sections in Math 311 class $\}$

$B = \{$ TAs in Math Dept. $\}$

(Section x) $\xmapsto{f}$ (TA of the section x) is a well-defined map from $A \to B$. e.g. f(Section 2) = (TA Fei Qi)

(TA y) $\xmapsto{g}$ (Section TA y is in charge of) is NOT a well-defined map from $B \to A$, b/c Fei Qi is in charge of two sections, namely H1 and 02. Not unique!

When A, B are sets of numbers, we call $f: A \to B$ a function.

Historically, functions were introduced and studied prior to maps

Example: $x \in \mathbb{R} \mapsto$ square root of x is NOT a well-defined function.

$x \in \mathbb{R} \mapsto$ positive square root of x is a well-defined function.

Think about why.

For a function $f: A \to B$,

- A is referred as its domain, B is referred as its codomain.
  - * Properties may change if the domain or codomain changes for the same rule of assignment!
- For $X \subseteq A$ subset, the collection of all elements in B that A maps to is called the image of X, denoted $f(X)$.

  Formally: $f(X) = \{ y \in B : \ y = f(x) \text{ for some } x \in X \}$

  Equivalently: $y \in f(X) \Leftrightarrow \exists x \in X, \ y = f(x)$
- For $Y \subseteq B$ subset, the collection of all elements in A that map into Y is called the preimage of Y, denoted $f^{-1}(Y)$

  Formally: $f^{-1}(Y) = \{ x \in A : \ f(x) \in B \}$

  Equivalently: $x \in f^{-1}(Y) \Leftrightarrow f(x) \in Y$.

Example: For $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Find

$$f(\mathbb{R}),\ f([0,\infty)),\ f([-9,16]),\ f([1,4])$$
$$f^{-1}(\mathbb{R}),\ f^{-1}([0,\infty)),\ f^{-1}([-9,16]),\ f^{-1}([1,4])$$

(Try it before looking at the answer next page).

Ans: $f(\mathbb{R})=\mathbb{R}$, $f([0,\infty))=[0,\infty)$. $f([-9,16])=[0,256]$. $f([1,4])=[1,16]$
$f^{-1}(\mathbb{R})=\mathbb{R}$. $f^{-1}([0,\infty))=\mathbb{R}$. $f^{-1}([-9,16])=[-4,4]$. $f^{-1}([1,4])=[-2,-1]\cup[1,2]$.

- $f: A\to B$ is called injective if

$$\forall x_1, x_2 \in A,\quad f(x_1)=f(x_2) \Rightarrow x_1=x_2.$$

Informally: If two numbers are mapped to the same number, then they are equal.

Equivalently: $\forall x_1, x_2 \in A$, $x_1\neq x_2 \Rightarrow f(x_1)\neq f(x_2)$

Informally, different numbers are mapped to different numbers.

- $f: A\to B$ is called surjective if

$$\forall y\in B,\ \exists x\in A,\ f(x)=y$$

Informally: Every number in the codomain is the image of some number in the domain.

Equivalently: $f(A)=B$.

Normally $f(A)$ is referred as range of f, denoted $\mathrm{Im} f$.

- f is bijective if f is both injective and surjective.

Example: $A, B\subseteq \mathbb{R}$. Find if $f: A\to B$, $f(x)=x^2$ is injective, surjective or bijective.

(1) $A=\mathbb{R}$, $B=\mathbb{R}$

(2) $A=\mathbb{R}$, $B=[0,\infty)$

(3) $A=\mathbb{R}$, $B=[-1,4]$.

(4) $A=\mathbb{R}$, $B=[1,4]$.

(5) $A=[0,\infty)$, $B=\mathbb{R}$.

(6) $A=(-\infty,0]$, $B=[0,\infty)$

(7) $A=[-1,4]$, $B=[1,16]$

(8) $A=[1,4]$, $B=[1,16]$

I wish this example convinces you that domain and codomain make a difference.

Try before looking at the next page.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Injective	×	×	×	×	✓	✓	×	✓
Surjective	×	✓	×	✓	×	✓	✓	✓
Bijective	×	×	×	×	×	✓	×	✓

Example: Let $f: A \to B$ be a function, $X \subseteq A$ be a subset.

Prove that $X \subseteq f^{-1}(f(X))$. Show by example that the inclusion can be proper, and prove that if f is injective, then $X = f^{-1}(f(X))$.

Sol'n: ① $x \in X \Rightarrow f(x) \in f(X)$

Y is introduced to avoid the confusion brought by notations.

Denote $Y = f(X) \subseteq B$, then $f(x) \in Y$

By def. of $f^{-1}(Y)$, we have $x \in f^{-1}(Y)$, which is $f^{-1}(f(X))$.

So we proved $x \in X \Rightarrow x \in f^{-1}(f(X))$.

Thus $X \subseteq f^{-1}(f(X))$ $\square$

② Take $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Let $X = [1,2]$. Compute $f^{-1}(f(X))$ to see.

$X \subsetneq f^{-1}(f(X))$ (Hint: Check the above examples for $f^{-1}(f(X))$)

③ We show $x \in f^{-1}(f(X)) \Rightarrow x \in X$ when f is injective.

$x \in f^{-1}(f(X)) \Rightarrow f(x) \in f(X) \subseteq B$

Denote $y = f(x) \in B$, then $y \in f(X)$

By def. of $f(X)$, $\exists\, x' \in X$, $f(x') = y$.

In general x' doesn't have to be x. But now that f is injective,

recall $y = f(x)$, so $f(x) = f(x') \Rightarrow x = x'$.

Since $x' \in X$, so $x = x' \Rightarrow x \in X$.

So we proved $x \in f^{-1}(f(X)) \Rightarrow x \in X$

Thus $f^{-1}(f(X)) \subseteq X$. $\square$.

Exercise: Let $f: A \to B$ be a function, $Y \subseteq B$ be a subset.
Prove that $f(f^{-1}(Y)) \subseteq Y$. Show by example that the inclusion can be proper, and prove that if f is surjective, then $f(f^{-1}(Y)) = Y$.

Exercise. $\forall X \subseteq A.\ X = f^{-1}(f(X)) \Leftrightarrow f$ is injective

Well-ordering principle: Every nonempty subset of $\mathbb{Z}_+$ has a smallest member.

Principle of Math Ind.: $P(1) \wedge (\forall n \in \mathbb{Z}_+, P(n) \Rightarrow P(n+1)) \Rightarrow (\forall n \in \mathbb{Z}_+, P(n))$

Proof: Assume $(\exists k \in \mathbb{Z}_+) \neg P(k)$.
Collect all such k's to form a subset of $\mathbb{Z}_+$.
Name it S. Well-ordering Principle $\Rightarrow$ S has a smallest elt, say k_0
i.e. $\forall n < k_0$, $P(n)$ is true. In particular, $P(k_0 - 1)$ is true.
From $(\forall n \in \mathbb{Z}_+) P(n) \Rightarrow P(n+1)$ with $P(k_0 - 1)$ is true
we know $P(k_0)$ is true, i.e. $k_0 \notin S$. Contradiction.
This proves $(\nexists k \in \mathbb{Z}_+) \neg P(k) \Leftrightarrow (\forall k \in \mathbb{Z}_+) P(k)$.

Example: $(\forall n \in \mathbb{Z}_+)$ $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ if $x \neq 1$.

If $n = 1$, LHS $= 1 + x$, RHS $= \dfrac{1 - x^2}{1 - x} = 1 + x$

Assume for $n = k$, LHS = RHS, i.e. $1 + x + \cdots + x^k = \dfrac{1 - x^{k+1}}{1 - x}$.

Want: $n = k+1$, LHS = RHS. i.e. Want $\underline{1 + x + \cdots + x^k} + x^{k+1} = \dfrac{1 - x^{k+2}}{1 - x}$

From ind. hypo: LHS $= \dfrac{1 - x^{k+1}}{1 - x} + x^{k+1} = \dfrac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x} = \dfrac{1 - x^{k+2}}{1 - x}$
$=$ RHS.

From principle of math. induction, LHS = RHS $\forall n \in \mathbb{Z}_+$. $\square$

Exercise: Prove that n^3+5n is divisible by 6. $\forall n \in \mathbb{Z}_+$.

<u>Recall</u>: Cartesian Product $A \times B = \{(a,b) : a \in A, b \in B\}$

Relation between A, B: any subset of $A \times B$.

Function: relation built on set of numbers, satisfying

$a \sim b_1,\ a \sim b_2 \Rightarrow b_1 = b_2 \in B.$

$f : A \to B$. If a is related to b, denote it by $f(a) = b$

the requirement means $a_1 = a_2 \in A \Rightarrow f(a_1) = f(a_2) \in B$

Given $f : A \to B$, let $M \subseteq A$, $N \subseteq B$.

Image of f on M is the subset $\{y = f(a) \in B : a \in M\}$ of B

denoted $f(M)$.

Preimage of f on N is the subset $\{a \in A : f(a) \in N\}$ of A.

<u>Example</u>: $f(x) = x^2$ gives a function $f : \mathbb{R} \to \mathbb{R}$

$f(\mathbb{R}) = \{x \geq 0\}$. $f([0,\infty)) = \{x \geq 0\}$ $f([2,4]) = \{4 \leq x \leq 16\}$

$f^{-1}([0,\infty)) = \mathbb{R}$ $f^{-1}(\mathbb{R}) = \mathbb{R}$. $f^{-1}([9,16]) = [-4,-3] \cup [3,4]$

$f^{-1}([-1,16]) = [-4,4]$ $f^{-1}((-\infty,0)) = \phi$

In general: $f: A \to B$ func.

$\Rightarrow X \subseteq f^{-1}(f(X))$. It might be a proper inclusion.

e.g. $f: x \mapsto x^2$. $X = [0, \infty)$. $f^{-1}(f([0,\infty))) = f^{-1}([0,\infty)) = \mathbb{R} \supset X$

Proof: $x \in X$, we should show $x \in f^{-1}(f(X))$

Recall $x \in f^{-1}(N)$ iff $f(x) \in N$.

$x \in f^{-1}(f(X)) \Leftrightarrow f(x) \in f(X)$, which is obvious if $x \in X$. (backward)

More formally: $x \in X \Rightarrow f(x) \in f(X) \Rightarrow x \in f^{-1}(f(X))$. (direct)

Q.E.D.

Exercise: Prove that for $Y \subseteq B$, $f[f^{-1}(Y)] \subseteq Y$.

In some cases, $X = f^{-1}(f(X))$. Guess the condition for f?

Ans: f is injective.

Proof: It suffices to show $f^{-1}(f(X)) \subseteq X$.

$x \in f^{-1}(f(X)) \Rightarrow f(x) \in f(X)$.

Recall: $a \in f(X) \Leftrightarrow \exists y \in X$, s.t. $f(y) = a$.

(Replace $a \mapsto f(x)$). $\Rightarrow \exists y \in X$, $f(y) = f(x)$

Recall: f is injective iff $f(x) = f(y)$ for some $x, y \in A \Rightarrow x = y$.

e.g. $x \mapsto x^2$ is injective as func. $[0,\infty) \to \mathbb{R}$

not injective as func. $\mathbb{R} \to \mathbb{R}$.

Inj $\Rightarrow y = x$. Since $y \in X$. $\Rightarrow x \in X$.

Exercise: Prove that for $Y \subseteq B$, $f(f^{-1}(Y)) = Y$ if f is surjective.